

A NOTE ON AFFINE LINKS

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ABSTRACT. We view the \tilde{A} -type affine braid group as a subgroup of the B -type braid group. We show that the \tilde{A} -type affine braid group surjects onto the A -type braid group and we detect the kernel of this surjection using Schreier's Theorem. We then describe an injection of the B -type braid group into the A -type braid group which allows us finally to give a definition of affine links, as closures of affine braids viewed as A -type braids after composing the above injections, and we prove that the two conditions of Markov are necessary and sufficient to get the same affine closure of any two affine braids.

1. INTRODUCTION

In [1] we define a tower of affine Temperley-Lieb algebras of type \tilde{A} , we define Markov conditions for traces relative to this tower, hence the affine notion of a Markov trace, and we show that there exists a unique Markov trace on the above tower. These results are obtained in a purely algebraic way, independently of their translation in topological terms, that is, in terms of invariants of links, however important. They were the goal of my work [2], in which the consequences in terms of invariants of links were explained. We describe below the braid groups of type A , B and \tilde{A} involved, both in an algebraic and a geometric way, we propose a definition of an affine link as the closure of an affine braid viewed as an A -type braid under an explicit injection, and we show that the two conditions of Markov are necessary and sufficient to get the same affine closure of any two affine braids.

2. ARTIN GROUPS AND BRAID GROUPS

Definitions and results of paragraphs 2 and 3 are taken mostly from [8].

Let S be a finite set.

Definition 2.1. *A Coxeter matrix over S is a square matrix $M = (m_{st})_{s,t \in S}$ such that*

- $m_{ss} = 1$,
- $m_{st} = m_{ts}$ for any s, t in S ,
- m_{st} belongs to $\{2, 3, 4, \dots, \infty\}$.

We present a Coxeter matrix by its Dynkin graph $\Gamma = \Gamma(M)$, which is a graph given by vertices and edges. Γ has S as a set of vertices, and for any non-equal two vertices s, t in S we have

- s, t are joined by an edge if $m_{s,t} = 3$,
- s, t are joined by a doubled edge if $m_{s,t} = 4$,
- s, t are joined by an edge labeled by $m_{s,t}$ if $m_{s,t} > 4$.

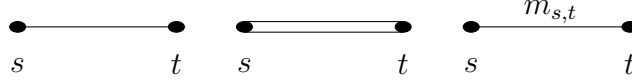


FIGURE 1. Edges

Definition 2.2. Let B_S be the set $\{\sigma_s; s \in S\}$. We call the pair (B, S) an Artin system of type Γ , where $B = B_\Gamma$ is the group given by generators and relations as follows: S is the set of generators with relations $\text{prod}(\sigma_s, \sigma_t : m_{st}) = \text{prod}(\sigma_t \sigma_s : m_{st})$, for any non-equal s, t in S with $m_{s,t} \neq \infty$.

We call B the Artin group of type Γ .

In our work we treat many kinds of Artin groups in which $m_{s,t} \leq 4$. The relations appearing in the definition are called "braid relations".

Let Γ be a Dynkin graph. Let (B, S) be the related Artin system. Take V to be the real vector space with \mathbb{R} -basis $\{e_s; s \in S\}$ which plays the role of the set of simple roots. The root system gives rise to simple reflections hence to a reflection group generated by those simple reflections, say W_S . By the natural linear representation of W_S we can realize it as a subgroup in $GL(V)$ (the group of endomorphisms of V). Let R be the set of reflections of W_S (the set of conjugates of simple reflections). Take r in R , since it is a reflection it fixes a hyper-plane in V , say H_r . In fact W_S acts freely on the complement of $\cup_{r \in R} H_r$ in V . By extending the action of W_S up to $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ we see that W_S acts freely on the complement of $\cup_{r \in R} \mathbb{C} \otimes_{\mathbb{R}} H_r$ in $V_{\mathbb{C}}$. We call this complement M_Γ . We set $N_\Gamma = M_\Gamma / W_S$.

Remark 2.3. Before stating the theorem, we have to notice that the above argument is valid in the case of finite W_S . For when the group W_S is infinite we have to replace V by $U \subset V$ (the so-called Tits cone), and to replace as well $V_{\mathbb{C}}$ by $(U + iV) \subset V_{\mathbb{C}}$. M_Γ is to be $(U + iV) - \cup_{r \in R} \mathbb{C} \otimes_{\mathbb{R}} H_r$. The action of W_S on M_Γ is free, and as above N_Γ is M_Γ modulo the action of W_S .

Definition 2.4. The braid group of Γ -type is $\pi_1(N_\Gamma)$, the fundamental group of the space N_Γ .

Theorem 2.5. (Brieskorn-Van der Lek). $\pi_1(N_\Gamma) \simeq B$.

Briefly: the Γ -type Artin group is given by a presentation, while the Γ -type braid group is a fundamental group. Although the Brieskorn-Van der Lek isomorphism is not canonical, we will not make the distinction in this work. We will call each of these groups a Γ -type braid group.

3. A-TYPE BRAID GROUPS

An A -type braid group with n generators is historically the first braid group. We give its presentation by generators and relations, then a geometrical one (by means of braids with $n + 1$ strands). Many interesting basic facts show the reasons for which it has such respectable position in the group theory, in addition to many other branches of mathematics: for example it has a faithful representation in $Aut(F_{n+1})$, the group of automorphisms of the free group with n generators; it has strong relations with link theory (here comes the well known Alexander theorem); in addition of its Garsiditude, in fact, it is the first group to be called a Garside group.

3.1. Presentations. Let $n \geq 1$ be an integer.

Definition 3.1. *The A -type braid group $B(A_n)$ with n generators is the group presented by a generator set $S = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ and the relations*

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ and $|i - j| \geq 2$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ where $1 \leq i \leq n - 1$.

Thus the related Dynkin diagram is

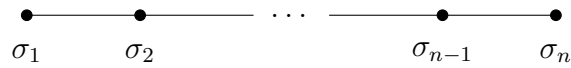


FIGURE 2. Γ_A

Let P_1, \dots, P_{n+1} be distinct points in the plane \mathbb{R}^2 . We define a rough braid on $n + 1$ strands to be an n -tuple $\beta = (b_1, \dots, b_n)$, where b_k is a path $b_k : [0, 1] \rightarrow \mathbb{R}$ such that

- For any k in $\{1, \dots, n + 1\}$ we have $b_k(0) = P_k$,
- For any k in $\{1, \dots, n + 1\}$ there exists a permutation $x = \theta(\beta) \in Sym_{n+1}$ such that $b_k(1) = P_{x(k)}$,
- For any non-equal k and l in $\{1, \dots, n + 1\}$, for all $t \in [0, 1]$ we have $b_k(t) \neq b_l(t)$.

By definition: two rough braids α and β are homotopic if there exists a continuous family of rough braids $\{\gamma_s\}_{s \in [0,1]}$ such that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. This is an equivalence relation.

Definition 3.2. *A braid on $n + 1$ strands is a homotopy class of rough braids on $n + 1$ strands.*

The well known geometric interpretation of the elements of $B(A_n)$ viewed as braids in the space is the following

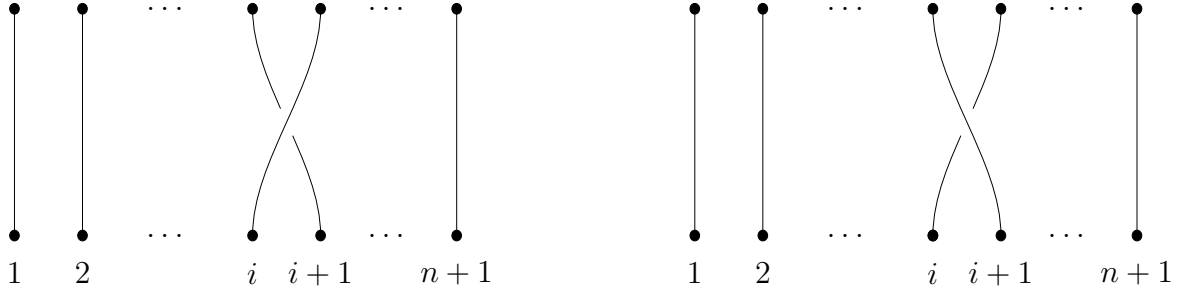


FIGURE 3. σ_i & σ_i^{-1}

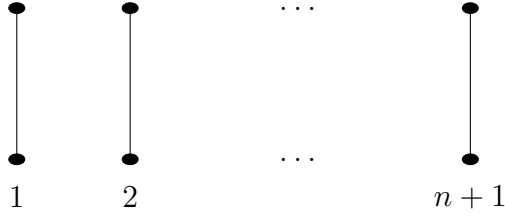
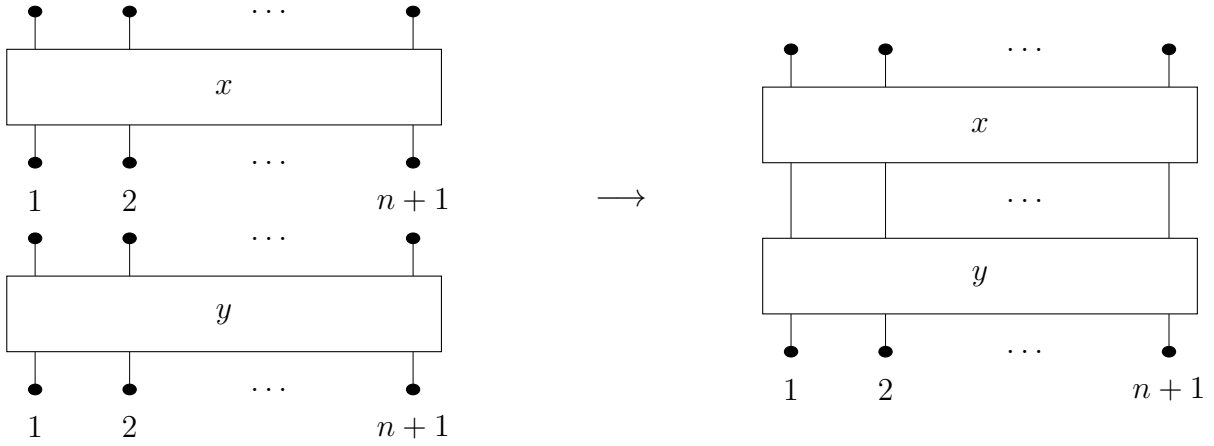


FIGURE 4. Id

We compose two braids in the way that one would expect, that is for any two braids X, Y the composed braid XY is the braid obtained by putting X at the top and Y at the bottom, welding the bottom end points of X with the upper ones of Y (the i -th with the i -th, $1 \leq i \leq n + 1$) as follows:

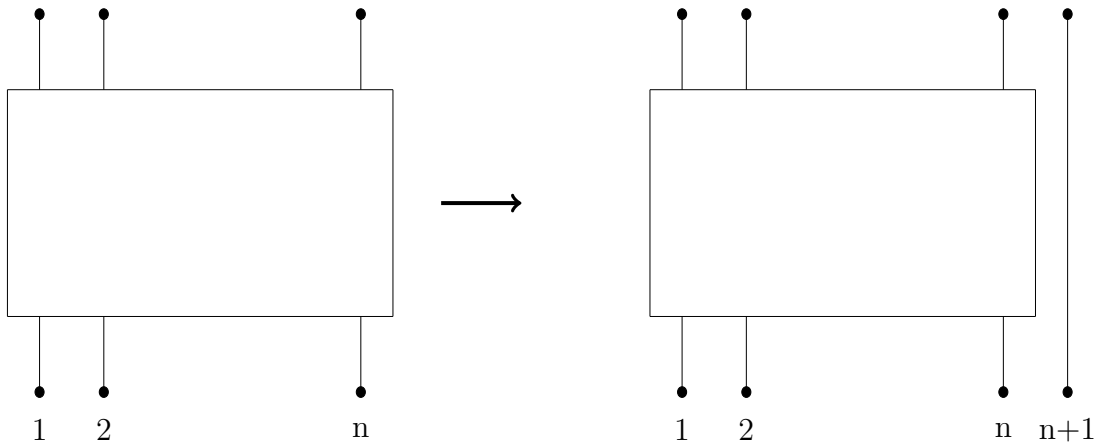

 FIGURE 5. $X, Y \rightarrow XY$

The natural embedding

$$x_{n-1} : B(A_{n-1}) \longrightarrow B(A_n)$$

$$\sigma_i \longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1,$$

can be realized geometrically by adding the $(n+1)$ -th strand


 FIGURE 6. x_{n-1}

3.2. A faithful representation. We can realize $B(A_n)$ as a subgroup of the group of automorphisms of the free group with $n+1$ generators via the "Artin representation", which will be briefly defined in what follows: Let F_{n+1} be the free group with $n+1$ generators x_1, \dots, x_{n+1} . Let $Aut(F_{n+1})$ be the group of automorphisms of F_{n+1} . For $1 \leq k \leq n$, we define t_k in $Aut(F_{n+1})$ as follows for $i \neq k, k+1$:

$$\begin{aligned} t_k : F_{n+1} &\longrightarrow F_{n+1} \\ x_i &\longmapsto x_i, \\ x_k &\longmapsto x_n^{-1} x_{k+1} x_k, \\ x_{k+1} &\longmapsto x_k. \end{aligned}$$

It is easy to show that the map $\rho : B(A_n) \longrightarrow Aut(F_{n+1})$, which sends σ_k to t_k , defines a representation of $B(A_n)$ in $Aut(F_{n+1})$ called the Artin representation.

Theorem 3.3. (*Artin*) *The representation ρ is faithful.*

4. B-TYPE BRAID GROUPS

The B -type braid group with $n+1$ generators $B(B_{n+1})$ plays a role in the theory of low dimension topological spaces, in addition to the fact that it is very useful in investigating the structure of the affine braid group which is the center of interest of this work in general. Definitions and results in this paragraph are taken from [6], where we can see more details about the group $B(B_{n+1})$.

4.1. Presentations.

Definition 4.1. *The B -type braid group with $n+1$ generators $B(B_{n+1})$ is the group presented by a generators set $\{\sigma_1, \sigma_2, \dots, \sigma_n, t\}$ and the relations*

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n$,
- $\sigma_i t = t \sigma_i$ when $2 \leq i \leq n$,
- $\sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1$.

The related Dynkin diagram is



FIGURE 7. $\Gamma_{B_{n+1}}$

Set $\phi_{n+1} = t\sigma_1 \dots \sigma_n$. Set $a_{n+1} = \phi_{n+1}\sigma_n\phi_{n+1}^{-1}$. We can see directly that $\phi_{n+1}\sigma_i\phi_{n+1}^{-1} = \sigma_{i+1}$ for all $1 \leq i \leq n-1$, with $\phi_{n+1}\sigma_n\phi_{n+1}^{-1} = a_{n+1}$. What is more, we have the following presentation:

Proposition 4.2. *$B(B_{n+1})$ is presented by the set of generators $S' = \{\sigma_1, \dots, \sigma_n, a_{n+1}, \phi_{n+1}\}$ together with the relations*

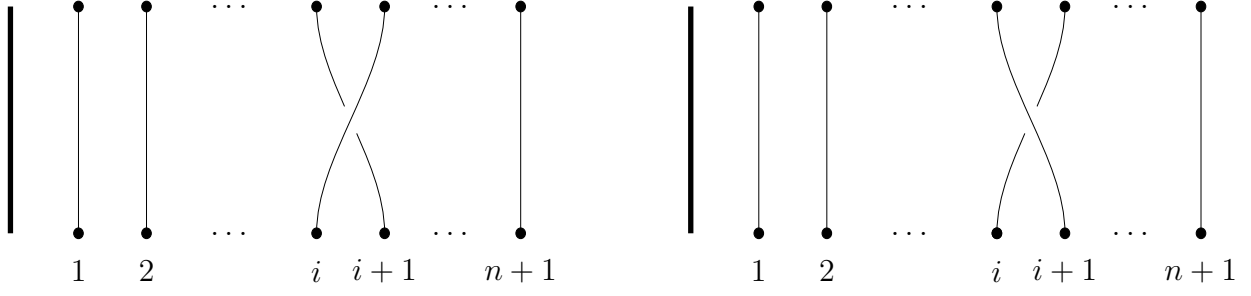
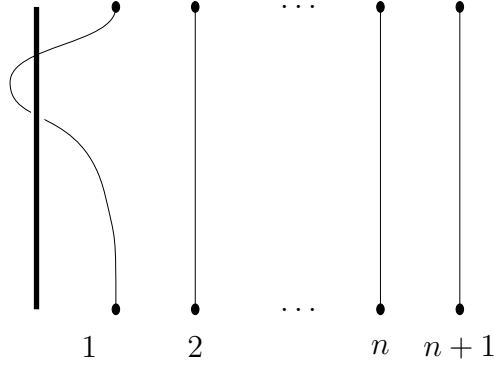
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \leq i \leq n-1$,
- $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ when $2 \leq i \leq n-1$,
- $\sigma_1 a_{n+1} \sigma_1 = a_{n+1} \sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1} \sigma_n = a_{n+1} \sigma_n a_{n+1}$,
- $\phi_{n+1}\sigma_i\phi_{n+1}^{-1} = \sigma_{i+1}$ when $1 \leq i \leq n-1$,
- $\phi_{n+1}\sigma_n\phi_{n+1}^{-1} = a_{n+1}$,
- $\phi_{n+1}a_{n+1}\phi_{n+1}^{-1} = \sigma_1$.

Notice that the relations involving ϕ_{n+1} are not braid relations, i.e., $(B(B_{n+1}), S')$ cannot be viewed as an Artin system.

Considering those relations we see directly that ϕ_{n+1} is acting by automorphisms on the normal subgroup of $B(B_{n+1})$ generated by $\{\sigma_1, \dots, \sigma_n, a_{n+1}\}$. More, this set $\{\sigma_1, \dots, \sigma_n, a_{n+1}\}$ with the first five systems of relations forms a presentation by generators and relations of this subgroup which will be the subject of the next section. The element ϕ_{n+1} generates a free subgroup of rank 1, which we denote by Φ_{n+1} . In other terms:

$$B(B_{n+1}) = \langle \sigma_1, \dots, \sigma_n, a_{n+1} \rangle_{B(B_{n+1})} \rtimes \Phi_{n+1}.$$

The geometric presentation of $B(B_{n+1})$ as braids is given in a very similar way of that of $B(A_n)$ but in $n+2$ strands, where the first strand remains point-wise fixed. The generators are presented as follows:

FIGURE 8. σ_i & σ_i^{-1} FIGURE 9. t

5. AFFINE BRAIDS: THE GROUP $B(\tilde{A}_n)$

The \tilde{A} -type affine braid group in $n + 1$ generators is the braid group under question in this work. Geometrically, one can see several presentations in the literature, among which we choose one which is compatible with our viewpoint on this group (as a base point for a better understanding of a special kind of links in the space). We show the strong connection between this group and the two groups mentioned above. The arrows in this section are well known, except for the surjection of $B(\tilde{A}_n)$ onto $B(A_n)$ which allows us to see $B(\tilde{A}_n)$ as a semi-direct product of $B(A_n)$ with a "huge" group, an infinitely generated free group (the semi-direct does not seem of a great use). In the other hand this surjection (and others induced by it) plays an essential role in [1]. While concerning $B(\tilde{A}_n)$ presentations, we define a new presentation (not far from the old one) called the parabolic-like presentation.

5.1. Presentations.

Definition 5.1. *The \tilde{A} -type braid group with $n+1$ generators $B(\tilde{A}_n)$ is the group presented by a set of generators $\{\sigma_1, \dots, \sigma_n, a_{n+1}\}$ together with the following defining relations*

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n - 1$,
- $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n - 1$,
- $\sigma_1 a_{n+1} \sigma_1 = a_{n+1} \sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1} \sigma_n = a_{n+1} \sigma_n a_{n+1}$.

Remark 5.2. *In the literature a_{n+1} in this definition is often called σ_{n+1} , but since we are interested in viewing $B(\tilde{A}_{n-1})$ as a subgroup of $B(\tilde{A}_n)$ for $2 \leq n$, there would be a confusion between a_n and σ_n when they are seen as elements in $B(\tilde{A}_n)$. Thus we consider the group $B(A_n)$ generated by $\{\sigma_1, \dots, \sigma_n\}$, then we "affinize" it by adding a_{n+1} .*

Remark 5.3. *It is not a coincidence that the generators here have the same symbols as the generators of $B(B_{n+1})$ in the second presentation of $B(B_{n+1})$. There is an obvious homomorphism $B(\tilde{A}_n) \longrightarrow B(B_{n+1})$, saying that this homomorphism is injective is equivalent to saying that in $B(B_{n+1})$ the subgroup generated by the elements $\sigma_1, \dots, \sigma_n, a_{n+1}$ is presented by generators and relations in the following way: it has for generating set $\{\sigma_1, \dots, \sigma_n, a_{n+1}\}$ together with the relations*

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n - 1$,
- $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n - 1$,
- $\sigma_1 a_{n+1} \sigma_1 = a_{n+1} \sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1} \sigma_n = a_{n+1} \sigma_n a_{n+1}$.

This is true, since ϕ_{n+1} acts on this very group by automorphisms, thus the relations between its generators do not add any other relation than the length-respecting braid relations already existing in the definition.

Proposition 5.4. *Let x be in $B(B_{n+1})$. Suppose that x is expressed as a word in the generators: $\sigma_1, \sigma_2, \dots, \sigma_n, t$. Then*

$$x \in B(\tilde{A}_n) \iff \text{the sum of the exponents of } t \text{ in } x \text{ is zero.}$$

Proof. Suppose that $x = u_1 t^{b_1} u_2 t^{b_2} u_3 \dots u_m t^{b_m} u_{m+1}$, where b_i is an integer and u_i is in $B(A_n)$ for all i .

We have $\phi_{n+1} = t\sigma_1 \dots \sigma_{n-1}\sigma_n$ by definition. Set $z^{-1} = \sigma_1 \dots \sigma_{n-1}\sigma_n$, that gives $t = \phi_{n+1}z$. We denote the action of ϕ_{n+1}^r on an element e in $B(\tilde{A}_n)$ by $[e]^r$, for any integer r . For example $\phi_{n+1}z = [z]^1\phi_{n+1}$.

$$\text{Now } t^{b_i} = \underbrace{\phi_{n+1}z\phi_{n+1}z\dots\phi_{n+1}z}_{b_i \text{ times}}, \text{ which is equal to } \phi_{n+1}^{b_i} \prod_{j=0}^{j=b_i} [z]^{j-b_i}.$$

$$\text{Set } Z_{b_i} = \prod_{j=0}^{j=b_i} [z]^{j-b_i}, \text{ which is in } B(\tilde{A}_n). \text{ Thus:}$$

$x = u_1 \phi^{b_1} Z_{b_1} u_2 \phi^{b_2} Z_{b_2} u_3 \dots u_m \phi^{b_m} Z_{b_m} u_{m+1}$. By pushing the " ϕ^{b_i} "s to the right (acting on the " u_i "s as well as on the " Z_{b_i} "s) we get the following expression of x :

$$x = \lambda \phi^{\left(\sum_{i=1}^{i=m} b_i\right)} \text{ where } \lambda \in \tilde{A}_n. \text{ This is the unique decomposition of } x \text{ in the semi-direct product } \langle \sigma_1, \sigma_2 \dots \sigma_n, a_{n+1} \rangle_{B(B_{n+1})} \rtimes \Phi_{n+1}.$$

Now x is in $B(\tilde{A}_n)$ if and only if $\sum_{i=1}^{i=m} b_i = 0$. The proposition follows. \square

Inspired by the injection $B(\tilde{A}_n) \hookrightarrow B(B_{n+1})$, we explain the geometric presentation we choose in this work. Actually, affine braids with $n+1$ generators could be viewed for example as cylindrical braids within a cylinder labeled by $n+1$ points on each of its circles, the strings of the braids are not allowed to make perfect rounds. We choose to view the affine braids as B -type braids: using the proposition above, the affine braids are B -type braids in which the number of positive rounds equals the number of negative rounds (around the fixed strand). The " σ "s are as presented above, while a_{n+1} and ϕ_{n+1} are the following braids:

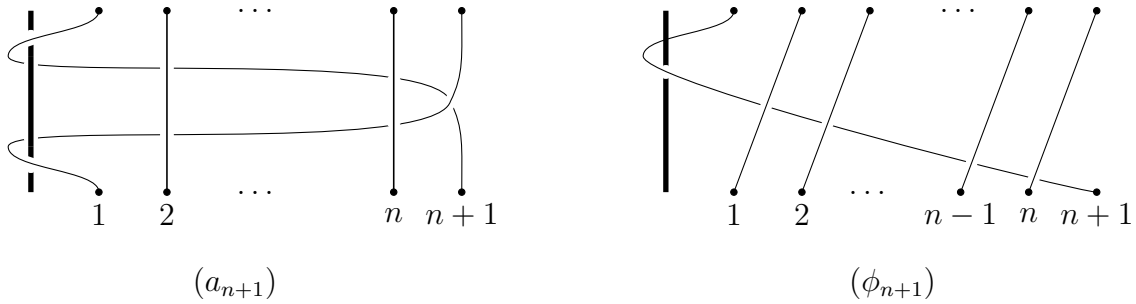


FIGURE 10.

Notice that ϕ_{n+1} is not an affine braid.

Proposition 5.5. *The following group homomorphism is injective*

$$\begin{aligned} F_n : B(\tilde{A}_{n-1}) &\longrightarrow B(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \quad \text{for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1} \end{aligned}$$

Proof. See 5.4: F_n is a restriction of the injection y_n to $B(\tilde{A}_n)$. \square

We give now a new presentation of $B(\tilde{A}_n)$, in which the defining relations are positive, and where F_n is obtained by simply adding one generator to those of $B(\tilde{A}_{n-1})$.

By definition $B(\tilde{A}_n)$ has $\{\sigma_1, \dots, \sigma_n, a_{n+1}\}$ as a set of generators together with the following defining relations:

- (1') $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- (2') $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n-1$,
- (3') $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n-1$,
- (4') $\sigma_1 a_{n+1} \sigma_1 = a_{n+1} \sigma_1 a_{n+1}$,
- (5') $\sigma_n a_{n+1} \sigma_n = a_{n+1} \sigma_n a_{n+1}$ for $n \geq 2$.

This presentation is to be called the formal presentation from now on. For the moment n is to be greater than or equal to 3. $B(\tilde{A}_2)$ is generated by σ_1, σ_2 and a_3 , we see that a_{n+1} can be seen as follows:

$$a_{n+1} = \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n.$$

Our aim is to show that $B(\tilde{A}_n)$ can be generated by $S' = \{\sigma_1, \dots, \sigma_n, a_3\}$ with defining relations:

- (1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- (2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n-1$,
- (3) $\sigma_1 a_3 \sigma_1 = a_3 \sigma_1 a_3$,
- (4) $\sigma_3 a_3 \sigma_3 = a_3 \sigma_3 a_3$,
- (5) $\sigma_i a_3 = a_3 \sigma_i$ when $4 \leq i \leq n$,
- (6) $\sigma_3 \sigma_2 a_3 \sigma_3 = \sigma_2 a_3 \sigma_3 \sigma_2$.

First we show that the formal presentation gives the new one, now $\{\sigma_1, \dots, \sigma_n, a_3\}$ generates $B(\tilde{A}_n)$ indeed, since

$$a_{n+1} = \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n, \quad \text{which gives} \quad a_3 = \sigma_3 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1}.$$

We see that (3) follows directly from the fact that a_3 and a_{n+1} are conjugate, while (5) could be seen to be valid geometrically or by a direct computation:

$$\begin{aligned}
 \text{for } 4 \leq i \leq n, \text{ we have : } \sigma_i a_3 &= \sigma_i \sigma_3 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1} = \sigma_3 \dots \sigma_n \sigma_{i-1} a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1} \\
 &= \sigma_3 \dots \sigma_n a_{n+1} \sigma_{i-1} \sigma_n^{-1} \dots \sigma_3^{-1} = \sigma_3 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1} \sigma_i \\
 &= a_3 \sigma_i.
 \end{aligned}$$

Now we treat (4):

$$\sigma_3 a_3 \sigma_3 = \sigma_3^2 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_4^{-1}.$$

$$\begin{aligned}
 \text{Thus, } a_3 \sigma_3 a_3 &= \sigma_3 \dots \sigma_n a_{n+1} \underbrace{\sigma_n^{-1} \dots \sigma_4^{-1} \sigma_3 \sigma_4 \dots \sigma_n}_{\sigma_n^{-1} \dots \sigma_4^{-1} \sigma_3 \sigma_4 \dots \sigma_n} a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1} \\
 &= \sigma_3 \dots \sigma_n a_{n+1} \sigma_3 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1} a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1} \\
 &= \sigma_3 \dots \sigma_n \sigma_3 \dots \sigma_{n-1} a_{n+1} \sigma_n a_{n+1} \sigma_{n-1}^{-1} \dots \sigma_3^{-1} \sigma_n^{-1} \dots \sigma_3^{-1} \\
 &= \sigma_3 \dots \sigma_n \sigma_3 \dots \sigma_{n-1} \sigma_n a_{n+1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1} \sigma_n^{-1} \dots \sigma_3^{-1} \\
 &= \sigma_3 \dots \sigma_n \sigma_3 \dots \sigma_{n-1} \sigma_n a_{n+1} \sigma_n \sigma_n^{-1} \dots \sigma_3^{-1} \sigma_n^{-1} \dots \sigma_4^{-1} \\
 &= \sigma_3 \dots \sigma_n \underbrace{\sigma_3 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1}}_{\sigma_3 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1}} a_{n+1} \sigma_n^{-1} \dots \sigma_4^{-1} \\
 &= \sigma_3 \dots \sigma_n \sigma_n^{-1} \dots \sigma_4^{-1} \sigma_3 \sigma_4 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_4^{-1} \\
 &= \sigma_3^2 \sigma_4 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_4^{-1} \\
 &= \sigma_3 a_3 \sigma_3.
 \end{aligned}$$

We see that (6) is equivalent to

$$\sigma_2 \sigma_3^{-1} a_3 \sigma_3 = \sigma_3^{-1} a_3 \sigma_3 \sigma_2 \quad \dots (6'').$$

But $\sigma_3^{-1} a_3 \sigma_3 = \sigma_4 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_4^{-1}$, and (6) follows.

Now we show that the new presentation gives the formal one.

We are reduced to show that the new presentation gives (3'), (4') and (5'), i.e., the formal relations which involve a_{n+1} have to be shown using the new relations with:

$$a_{n+1} = \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n.$$

We start by dealing with (3'): let $3 \leq i \leq n-1$. We compute:

$$\begin{aligned} \sigma_i a_{n+1} &= \sigma_i \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n = \sigma_n^{-1} \dots \sigma_3^{-1} \underbrace{\sigma_{i+1} a_3}_{(5)} \sigma_3 \dots \sigma_n \\ &= \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_{i+1} \sigma_3 \dots \sigma_n = \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n \sigma_i \\ &= a_{n+1} \sigma_i. \end{aligned}$$

$$\begin{aligned} \text{Moreover, } \sigma_2 a_{n+1} &= \sigma_2 \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n = \sigma_n^{-1} \dots \underbrace{\sigma_2 \sigma_3^{-1} a_3 \sigma_3}_{(6'')} \dots \sigma_n = \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \sigma_2 \dots \sigma_n \\ &= a_{n+1} \sigma_2. \end{aligned}$$

Hence, $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n-1$. Thus, (3') is proved.

Since a_{n+1} and a_3 are conjugate, (3) gives directly (4').

Now we want to show that $\underbrace{\sigma_n a_{n+1} \sigma_n}_{:=x} = \underbrace{a_{n+1} \sigma_n a_{n+1}}_{:=y}$.

$$x = \sigma_{n-1}^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n^2,$$

$$\begin{aligned} y &= \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_n = \sigma_n^{-1} \dots \sigma_3^{-1} a_3 \sigma_n^{-1} \dots \sigma_4^{-1} \sigma_3 \sigma_4 \dots \sigma_n a_3 \sigma_3 \dots \sigma_n \\ &= \sigma_n^{-1} \dots \sigma_3^{-1} \sigma_n^{-1} \dots \sigma_4^{-1} \underbrace{a_3 \sigma_3 a_3}_{(4)} \sigma_4 \dots \sigma_n \sigma_3 \dots \sigma_n = \sigma_n^{-1} \dots \sigma_3^{-1} \sigma_n^{-1} \dots \sigma_4^{-1} \sigma_3 a_3 \sigma_3 \sigma_4 \dots \sigma_n \sigma_3 \dots \sigma_n \\ &= \sigma_{n-1}^{-1} \dots \sigma_3^{-1} \sigma_n^{-1} \dots \sigma_3^{-1} \sigma_3 a_3 \sigma_3 \sigma_4 \dots \sigma_n \sigma_3 \dots \sigma_n = \sigma_{n-1}^{-1} \dots \sigma_3^{-1} a_3 \sigma_n^{-1} \dots \sigma_4^{-1} \sigma_3 \sigma_4 \dots \sigma_n \sigma_3 \dots \sigma_n \\ &= \sigma_{n-1}^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1} \sigma_3 \dots \sigma_n = \sigma_{n-1}^{-1} \dots \sigma_3^{-1} a_3 \sigma_3 \dots \sigma_{n-1} \sigma_n^2 \\ &= x. \end{aligned}$$

Finally, $F_n(a_3) = F_n(\sigma_3 \dots \sigma_{n-1} a_n \sigma_{n-1}^{-1} \dots \sigma_3^{-1}) = \sigma_3 \dots \sigma_{n-1} F_n(a_n) \sigma_{n-1}^{-1} \dots \sigma_3^{-1}$. This is equal to $\sigma_3 \dots \sigma_n a_{n+1} \sigma_n^{-1} \dots \sigma_3^{-1}$, thus to a_3 .

Now F_n would have the following form with the new presentation:

$$\begin{aligned} F_n : B(\tilde{A}_{n-1}) &\longrightarrow B(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_3 &\longmapsto a_3 \end{aligned}$$

Notice that we could have started with the group $B(\tilde{A}_1)$, which is a free group in two letters σ_1 and a_3 , with a change in the sixth relation. On the other hand, it is obvious that $(B(\tilde{A}_{n+1}), S')$ is not an Artin System.

5.2. $B(A_n)$ as a quotient of $B(\tilde{A}_n)$. Now we consider the element e in $B(\tilde{A}_n)$ given as

$$e = a_{n+1}^{-1} \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n = a_{n+1}^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1^{-1}.$$

Let N_e be the normal subgroup of $B(\tilde{A}_n)$ generated by e . Consider the quotient $Q = B(\tilde{A}_n)/N_e$.

Lemma 5.6. *In $B(\tilde{A}_n)$ (and in $B(A_n)$ as well), the element $b := \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n$ verifies the following relations:*

- 1) $\sigma_i b = b \sigma_i$ for $2 \leq i \leq n-1$,
- 2) $\sigma_n b \sigma_n = b \sigma_n b$,
- 3) $\sigma_1 b \sigma_1 = b \sigma_1 b$.

Proof. 1)

$$\begin{aligned} \sigma_i b &= \sigma_i \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n = \sigma_n^{-1} \dots \underbrace{\sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1}}_{\sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}} \dots \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n \\ &= \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \dots \underbrace{\sigma_{i+1} \sigma_i \sigma_{i+1}}_{\sigma_i \sigma_{i+1} \sigma_i} \dots \sigma_1 = b \sigma_i. \end{aligned}$$

3)

$$\begin{aligned} &\sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n \sigma_1 \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n \\ &= \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \underbrace{\sigma_1 \sigma_2 \sigma_1}_{\sigma_2 \sigma_1 \sigma_2} \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n = \sigma_1^2 \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_3^{-1} \sigma_2 \sigma_3 \dots \sigma_{n-1} \sigma_n = \sigma_1 b \sigma_1. \end{aligned}$$

In the same way we deal with (2), hence the proof is done. □

We see directly that, when replacing b by a_{n+1} in this lemma, we get the defining relations of $B(\tilde{A}_n)$ in which a_{n+1} is involved. Now the group Q is generated by the set $\{\sigma_1, \dots, \sigma_n, a_{n+1}\}$ with the defining relations :

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n - 1$,
- $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n - 1$,
- $\sigma_1 a_{n+1} \sigma_1 = a_{n+1} \sigma_1 a_{n+1}$ and $\sigma_n a_{n+1} \sigma_n = a_{n+1} \sigma_n a_{n+1}$. for $n \geq 2$,
- $a_{n+1} = \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n$.

Theorem 5.7. *The map $f : Q \longrightarrow B(A_n)$ defined by*

$$\begin{aligned} \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n, \\ a_{n+1} &\longmapsto \sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n, \end{aligned}$$

is a group isomorphism.

Proof. By the lemma we see that f is a homomorphism, and it is surjective. Moreover, the following map:

$$\begin{aligned} g : B(A_n) &\longrightarrow Q \text{ given by} \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n, \end{aligned}$$

is a group homomorphism, surjective indeed, for $\sigma_n^{-1} \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1 \sigma_2 \dots \sigma_{n-1} \sigma_n$ is sent to itself, hence to a_{n+1} .

Obviously: $f \circ g = Id_{B(A_n)}$ and $g \circ f = Id_Q$, so f is indeed an isomorphism with $f^{-1} = g$. \square

With the notation as above:

$$B(\tilde{A}_n) \cong B(A_n) \ltimes N_e$$

Clearly this surjection respects the inclusion F_n , in other terms the following diagram commutes:

$$\begin{array}{ccc} B(\tilde{A}_{n-1}) & \xleftarrow{F_n} & B(\tilde{A}_n) \\ \downarrow \beta_{n-1} & & \downarrow \beta_n \\ B(A_{n-1}) & \xleftarrow{x_n} & B(A_n) \end{array}$$

5.3. On the structure of $B(\tilde{A}_n)$: Schreier's theorem.

Lemma 5.8. *Let G be a free group with $S = \{F_0, F_1, \dots, F_n\}$ as a set of free generators, let N be the subgroup:*

$$N = \left\{ F_{i_1}^{\epsilon_1} \dots F_{i_j}^{\epsilon_j} \dots F_{i_m}^{\epsilon_m}; \sum \epsilon_j = 0 \right\}$$

That is the normal group of words in which the number of positive letters is equal to the number of negative letters.

Then, N is an infinitely generated free group over the letters $F_0^j F_i F_0^{-(j+1)}$ where j is an arbitrary integer .

Proof. By Schreier's theorem we see that N is a free group. Now we apply Schreier's algorithm to show that $F_0^j F_i F_0^{-(j+1)}$ could be viewed as free generators, to do so we consider the following surjection:

$$G \longrightarrow \langle F_0 \rangle \quad \text{given by:}$$

$$F_i \mapsto F_0 \text{ for } 1 \leq i \leq n,$$

where $\langle F_0 \rangle$ is to be the subgroup of G generated by F_0 , that is the free group with one generator F_0 . It is obvious that N is the kernel of this homomorphism (we could actually choose any of the letters of G to generate a range in order to have N as a kernel), hence

$$G \cong \langle F_0 \rangle \rtimes N.$$

So we can consider the set $\{F_0^j; j \in \mathbb{Z}\}$ as a full set of representatives of right cosets of N in G , thus it could be considered as a "Schreier's system". Now we define the following map:

$$\phi : G \longrightarrow \langle F_0 \rangle \quad \text{given by:}$$

$$\phi(x) = F_0^k \text{ when } x \in NF_0^k.$$

By Schreier's theorem: $gs\phi(gs)^{-1}$ are free generators of N , where g runs over the "Schreier's system" and s runs over the set of free generators of G .

The element gs is of the form $F_0^j F_i$, while $\phi(gs)^{-1}$ is of the form:

$$\phi(F_0^j F_i)^{-1} = \phi(F_i^{-1} F_0^{-j}) = \phi(\underbrace{F_i^{-1} F_0^{-j} F_0^{j+1}}_{\in N} F_0^{-(j+1)}) = F_0^{-(j+1)}.$$

Thus $gs\phi(gs)^{-1}$ is of the form $F_0^j F_i F_0^{-(j+1)}$ where $0 \leq i \leq n$. The lemma follows. \square

Now the following diagram commutes:

$$\begin{array}{ccc}
 B(\tilde{A}_n) & \xrightarrow{f} & B(A_n) \\
 \downarrow i_n & \nearrow \alpha_n & \\
 B(B_{n+1}) & &
 \end{array}$$

We can see that $\ker(f) = \ker(\alpha_n) \cap i_n(B(\tilde{A}_n))$. But $N_e = \ker(f)$, so we are reduced to understanding the structure of $\ker(\alpha_n)$. Set $G' := \ker(\alpha_n)$. From [4], section 5 and proposition 16, we get:

Lemma 5.9. [4] *With the above notations, G' is a free subgroup of $B(B_{n+1})$ generated by the free generators F_i , where $0 \leq i \leq n$, $F_i = \sigma_i \dots \sigma_1 t \sigma_1^{-1} \dots \sigma_i^{-1}$ and $F_0 = t$.*

An element x is in N_e if and only if it is a word in G' and it is in $B(\tilde{A}_n)$, but since any element of $B(B_{n+1})$ is in $B(\tilde{A}_n)$ if and only if the sum of exponents of t is zero in an (every) expression of it, so we can apply the last lemma, taking G' for G . Then N_e is to be N , so our group N_e is an infinitely generated free group.

5.4. $B(A_{n+1})$, $B(B_n)$ and $B(\tilde{A}_n)$, arrows.

In what follows we show the net of arrows between the three types of braid groups mentioned above. We are interested in investigating which arrows among those do respect the injections between groups of a given type in different number of generators. Roughly speaking: the arrows should be thought of as arrows defined over the "towers of groups", precisely the towers come from the injections between groups of the same types.

Consider x_n , the injection $B(A_{n-1}) \hookrightarrow B(A_n)$ mentioned in 1.2.1. Geometrically $B(B_n)$ embeds into $B(B_{n+1})$ by adding the $n+1$ -th stand, that is:

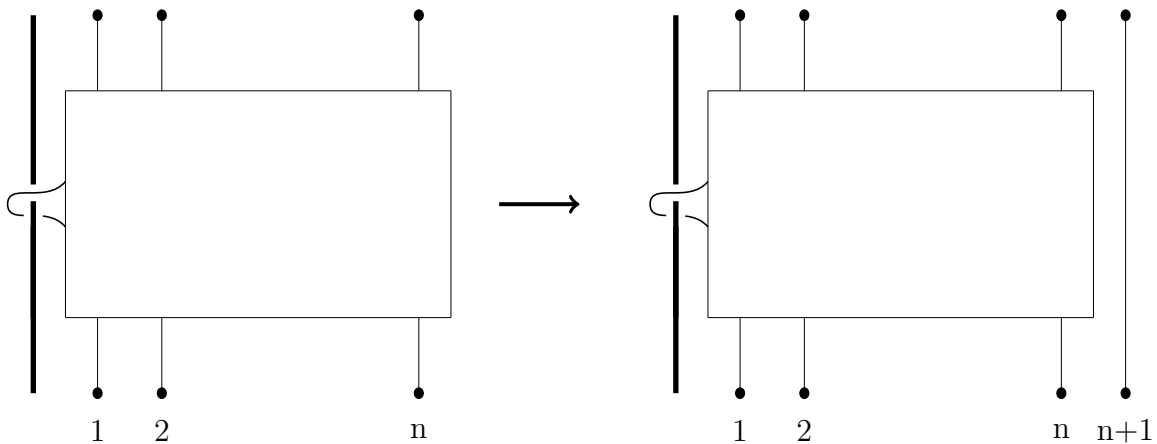


FIGURE 11. y_n

Let y_n be the injection $B(B_n) \hookrightarrow B(B_{n+1})$.

$B(A_n)$ injects in $B(B_{n+1})$ by sending σ_i to σ_i for $1 \leq i \leq n-1$, let us call this injection z_n . Take T to be the normal subgroup in $B(B_{n+1})$ generated by t , that is the subgroup generated by xtx^{-1} for $x \in B(B_{n+1})$. Obviously $B(B_{n+1})/T = B(A_n)$. in other words we have the following exact sequence $1 \rightarrow T \rightarrow B(B_{n+1}) \rightarrow B(A_n) \rightarrow 1$. Call α_n the surjection $B(B_{n+1}) \twoheadrightarrow B(A_n)$.

Geometrically $B(A_n)$ injects into $B(B_{n+1})$ by adding the first (fixed) strand, while $B(B_{n+1})$ surjects onto $B(A_n)$ by removing the very same strand

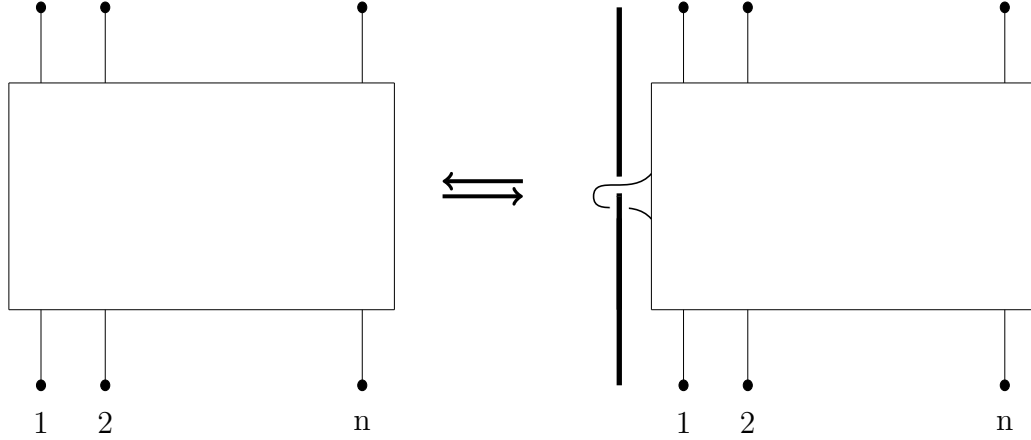


FIGURE 12. $B(A_n) \leftrightarrow B(B_{n+1})$

We get the following commutative diagrams:

$$\begin{array}{ccc}
 B(B_n) & \xleftarrow{y_n} & B(B_{n+1}) \\
 \uparrow z_{n-1} & & \uparrow z_n \\
 B(A_{n-1}) & \xleftarrow{x_n} & B(A_n)
 \end{array}
 \quad (1)$$

$$\begin{array}{ccc}
 B(B_n) & \xleftarrow{y_n} & B(B_{n+1}) \\
 \downarrow \alpha_{n-1} & & \downarrow \alpha_n \\
 B(A_{n-1}) & \xleftarrow{x_n} & B(A_n)
 \end{array}
 \quad (2)$$

Diagram 1 commutes obviously, while for diagram 2 it is clear that

$$\begin{aligned}
 x_n \alpha_{n-1}(\sigma_i) &= \alpha_n y_n(\sigma_i) \text{ for } 1 \leq i \leq n-1, \\
 x_n \alpha_{n-1}(t) &= 1 = \alpha_n y_n(t).
 \end{aligned}$$

The embedding of $B(\tilde{A}_{n-1})$ into $B(\tilde{A}_n)$ is not as obvious as the other two embeddings. Consider the injection $y_n : B(B_n) \hookrightarrow B(B_{n+1})$ (see [5]), which sends a_n to

$$t\sigma_1 \dots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1} \dots \sigma_1^{-1}t^{-1},$$

which is equal to

$$\begin{aligned} t\sigma_1 \dots \sigma_{n-2}\sigma_{n-1} \underbrace{\sigma_n \sigma_{n-1} \sigma_{n-1}^{-1} \sigma_n^{-1}}_1 \sigma_{n-2}^{-1} \dots \sigma_1^{-1}t^{-1} &= t\sigma_1 \dots \sigma_{n-2} \underbrace{\sigma_{n-1} \sigma_n \sigma_{n-1}^{-1}} \sigma_{n-1}^{-1} \sigma_n^{-1} \sigma_{n-2}^{-1} \dots \sigma_1^{-1}t^{-1} \\ &= t\sigma_1 \dots \sigma_{n-2} \sigma_n \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \sigma_n^{-1} \sigma_{n-2}^{-1} \dots \sigma_1^{-1}t^{-1} \\ &= \sigma_n \underbrace{t\sigma_1 \dots \sigma_{n-1} \sigma_n \sigma_{n-1}^{-1} \dots \sigma_1^{-1}t^{-1}}_{a_{n+1}} \sigma_n^{-1}. \end{aligned}$$

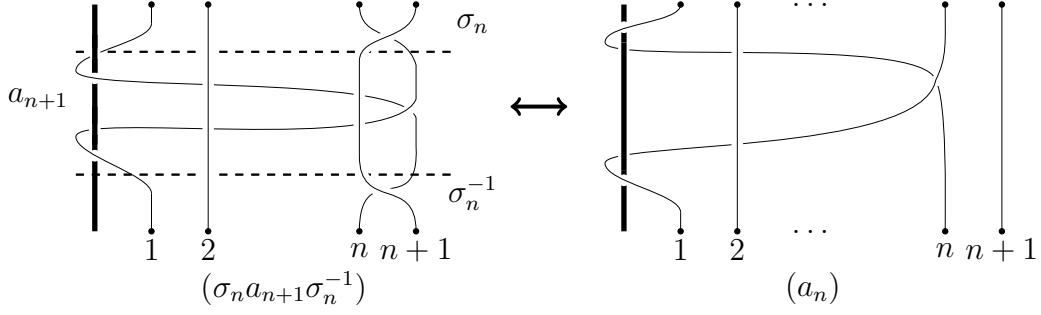
In other terms $y_n(a_n) = a_{n+1}$ which is in $B(\tilde{A}_n)$, in other terms the restriction of y_n to $B(\tilde{A}_{n-1})$ is equal to F_n as defined in Proposition 5.5, thus, this restriction is injective:

$$\begin{aligned} F_n : B(\tilde{A}_{n-1}) &\longrightarrow B(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1}. \end{aligned}$$

Set I_n to be the injection $B(A_n) \hookrightarrow B(\tilde{A}_n)$. Set i_n to be the injection $B(\tilde{A}_n) \hookrightarrow B(B_{n+1})$. We have the following commutative diagram:

$$\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ B(B_n) & \xleftarrow{y_n} & & & B(B_{n+1}) \\ & \nwarrow i_{n-1} & & \nearrow i_n & \\ & & B(\tilde{A}_{n-1}) & \xrightarrow{F_n} & B(\tilde{A}_n) \\ & \nearrow I_{n-1} & & \nwarrow I_n & \\ & & B(A_{n-1}) & \xleftarrow{x_n} & B(A_n) \\ & \nwarrow z_{n-1} & & \nearrow z_n & \\ & & & & \end{array}$$

Geometrically, we realize in $B(\tilde{A}_n)$ the generators σ_i of $B(\tilde{A}_{n-1})$ (for $1 \leq i \leq n-1$) in the natural way, while concerning a_n we see that

FIGURE 13. F_n

Recall that $B(B_n)$ is generated by $\{\sigma_1, \dots, \sigma_n, a_n, \phi_n\}$. Now we consider ϕ_n as an automorphism of $B(\tilde{A}_{n-1})$. We call ϕ_n the Dynkin automorphism of order n , since it generates a subgroup of $\text{Aut}(B(\tilde{A}_{n-1}))$ of order n . It shifts the generators of the Dynkin diagram one step counter clockwise ($\sigma_1 \mapsto \sigma_2 \mapsto \dots \mapsto \sigma_{n-1} \mapsto a_n \mapsto \sigma_1$). In order to simplify we call it the Dynkin automorphism, referring to it by ϕ when there is no ambiguity.

On the other hand we see that in $B(\tilde{A}_n)$ for $2 \leq i \leq n-1$:

$$\begin{aligned} \sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1} \sigma_i &= \sigma_{i-1} \sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1}, \\ \sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1} \sigma_1 &= a_n \sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1}, \\ \sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1} a_n &= \sigma_{n-1} \sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1}. \end{aligned}$$

The last equality comes from the fact that $a_{n+1} a_n = \sigma_n^{-1} a_n \sigma_n a_n = \sigma_n^{-1} \sigma_n a_n \sigma_n$, which is equal to $a_n \sigma_n = \sigma_n a_{n+1}$. Hence $\sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1}$ acts on the elements of $B(\tilde{A}_{n-1})$ exactly the way as ϕ_n^{-1} does in $B(B_n)$.

Definition 5.10. In $B(\tilde{A}_{n-1})$ we call $\sigma_n \sigma_{n-1} \dots \sigma_1 a_{n+1}$ the dominating element, and we denote it by D_n . When there is no ambiguity we call it D .

The following diagram commutes

$$\begin{array}{ccc} & B(B_n) & \\ & \uparrow i_{n-1} & \searrow f_n \\ B(\tilde{A}_{n-1}) & \xleftarrow{F_n} & B(\tilde{A}_n) \end{array}$$

$$\begin{aligned}
 \text{where } f_n : B(B_n) &\longrightarrow B(\tilde{A}_n) \\
 \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\
 a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1} \\
 \phi_n &\longmapsto D_n^{-1}.
 \end{aligned}$$

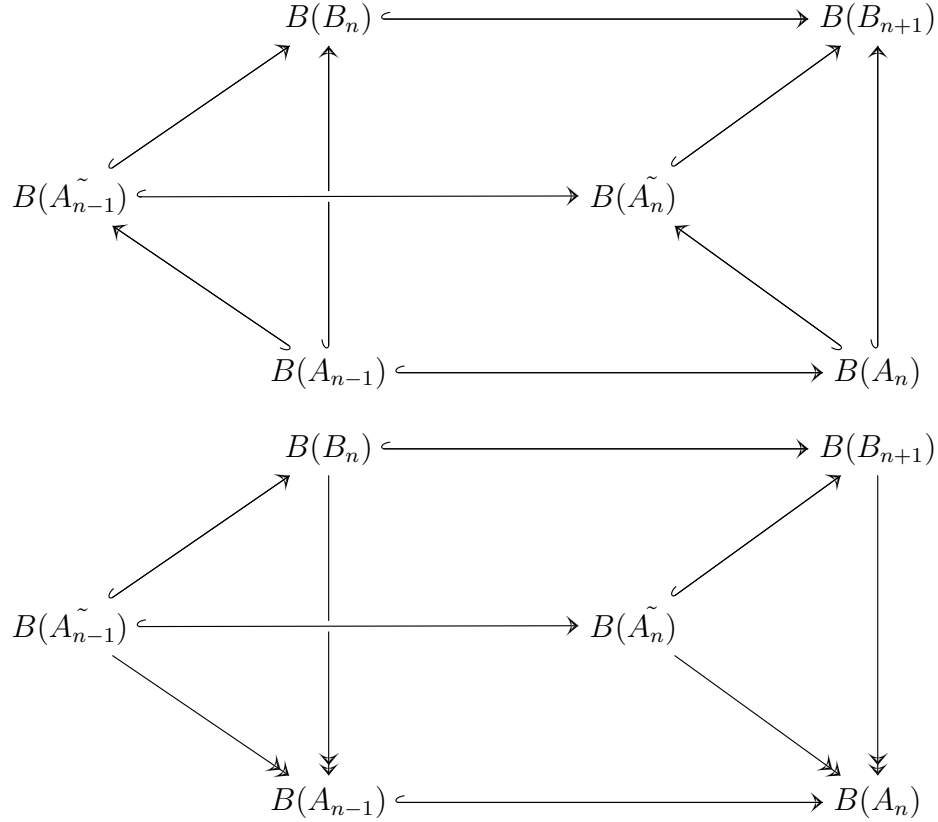
We consider the group $B(\tilde{A}_n)$ modulo the action of $\langle \phi_{n+1} \rangle_{\text{Aut}(B(\tilde{A}_n))}$ (the subgroup of $\text{Aut}(B(\tilde{A}_n))$ of order $n+1$ generated by the Dynkin automorphism). This group is isomorphic to the free group in one letter.

We have seen that $B(\tilde{A}_n)$ surjects onto $B(A_n)$. Call this surjection β_n , we get the following diagram:

$$\begin{array}{ccccc}
 B(\tilde{A}_{n-1}) & \xleftarrow{F_n} & & & B(\tilde{A}_n) \\
 \downarrow i_{n-1} & \searrow \beta_{n-1} & & & \swarrow \beta_n \downarrow i_n \\
 & B(A_{n-1}) & \xleftarrow{x_n} & B(A_n) & \\
 \nearrow \alpha_{n-1} & & & & \nwarrow \alpha_n \\
 B(B_n) & \xleftarrow{y_n} & & & B(B_{n+1})
 \end{array}$$

We have $\alpha_{n-1}i_{n-1}(a_{n-1}) = \alpha_{n-1}(t\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}t^{-1}) = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}$, which is equal to $\beta_{n-1}(a_{n-1})$. Hence $\alpha_{n-1}i_{n-1} = \beta_{n-1}$. But we know already that $i_n F_n = y_n i_{n-1}$ and $x_n \alpha_{n-1} = \alpha_n y_n$. Hence $x_n \beta_{n-1} = \beta_n F_n$.

Finally we present the arrows shown earlier by the following two diagrams:



5.5. Affine links and closures.

In what follows we give some definitions and basic results in the theory of links, the classical well known results concerning invariants of links are to be mentioned briefly. The aim is to define the concept of "affine" links, defining dual concepts and conventions to those in the classical theory. Most of theorems and results here are well explained in the literature, and we will not give details.

Let C_i be a circle in \mathbb{R}^3 , where $1 \leq i \leq n$. Let $C^n := \cup_i C_i$ be the disjoint union of those n circles. We call C^n a rough link. Take the isotopy class of C^n , call it C , we call C a circle link or simply a link. We consider here only the piecewise linear links. If we orient the circles forming C , we say that it is an oriented link. Notice that orienting a circle is independent of orienting another one, since they do not intersect. Inverting the orientation of one of the circles gives a different oriented link.

Roughly speaking, the problem of finding an invariant for the set of links in \mathbb{R}^3 is to give names to links, in such way that any two links which have the same "shape" (i.e., we can arrive to one from the other by pulling and pushing the circles forming a link without cutting, adding or omitting any of the circles) have the same name.

We recall the results of Alexander and Markov concerning braids and links.

Consider a braid b , that is: a an element of $B(A_n)$ for some $1 \leq n$. As we have $n + 1$ strands with $n + 1$ points at the top (the same at the bottom), a path from a point (say the i -th point) at the top to the i -th at the bottom makes the i -th strand turn into a deformation of a circle in \mathbb{R}^3 , repeating the same step with all the points (using non crossing paths) gives a union of disjoint deformed circles, hence a link. Thus we have defined a mapping from $\bigcup_{1 \leq i} B(A_i)$ into the set of links in \mathbb{R}^3 . We call the image of b : the closure of b , denoted by \hat{b} .

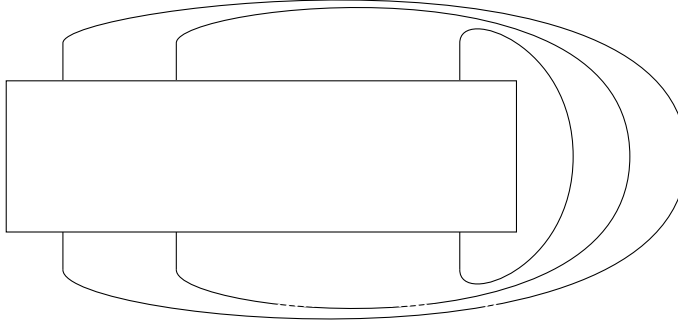


FIGURE 14.

We are interested in a map in the opposite direction.

Theorem 5.11. (Alexander) Suppose that C is an oriented link, then there exists an integer $1 \leq n$ and b in $B(A_n)$ such that $\hat{b} = C$.

In other terms, set OK to be the set of all oriented links in \mathbb{R}^3 . Then the following map is surjective:

$$\bigcup_{1 \leq i} B(A_i) \longrightarrow OK$$

$$b \longmapsto \hat{b}.$$

Let us present the main idea of the proof. Take any link C . Suppose it is the union of n deformed circles. We project it on \mathbb{R}^2 respecting the crossing points (the concept of positive and negative crossing makes it doable). We take any point in \mathbb{R}^2 , say P . We take an orientation of every circle (by arbitrary orientation of those circles we get all the possible orientations of C). The point P defines negative and positive rounds, say that the negative, for example, can be shifted to the 'right' of P the positive on the 'left', hence we arrive to some presentation of C as the following

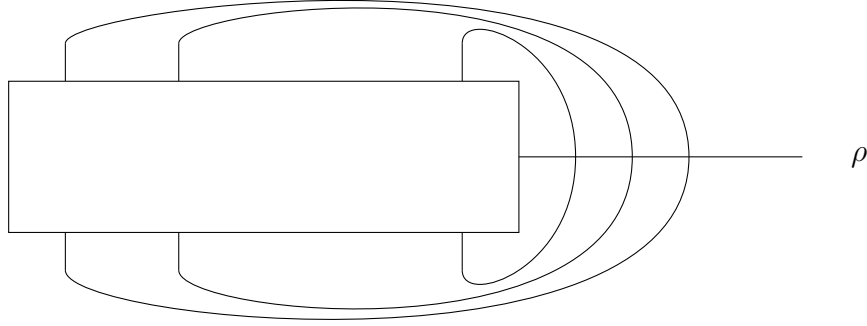


FIGURE 15.

Then we can cut the circle on the axis ρ , getting a braid whose closure is C .

The next question is, obviously: when do two braids have the same closure? The answer is a theorem of Markov – actually it was announced by Markov himself, finally proven by Birman.

Theorem 5.12. *(Markov) Two braids have the same closure if and only if there exists an integer $1 \leq n$ such that: starting from one of the two braids we can arrive to the other by a finite number of transformations of the two following types:*

- $ab \leftrightarrow ba$ where a and b are in $B(A_n)$,
- $x \leftrightarrow x\sigma_n$ where x is in $B(A_{n-1})$.

After this theorem a very elegant answer would be to find a family of applications t_{n+1} defined over $B(A_n)$ such that for all $1 \leq n$:

- $t_n(ab) = t_n(ba)$ for all a and b in $B(A_{n-1})$,
- $t_{n+1}(x\sigma_n) = t_n(x)$ for any x in $B(A_{n-1})$.

The answer given by Jones was exactly of this form. Here we attempt to define an "affine link" as a result of closing an affine braid, where we mean by affine braid an element of an \tilde{A} -type braid group. This task is not as evident as for the A -type braid group, for we have many geometrical presentations of \tilde{A} -type braid groups. A choice must be made here, this is what we are about to do in the rest of this section.

In order to simplify we call an oriented link simply a link in \mathbb{R}^3 (in the literature S^3 is often used).

We call a B -braid any element in a given B -type braid. Clearly any affine braid is a B -braid, which has a presentation as a cylindrical braid which could be closed at least in two ways. Here we view B -braids as we did above, braids with one fixed strand. Now we consider the following application:

$$\begin{aligned} {}_nI : B(B_n) &\longrightarrow B(A_n) \\ \sigma_i &\mapsto \sigma_{i+1} \text{ for } 1 \leq i \leq n-1 \\ t &\mapsto \sigma_1^2. \end{aligned}$$

It is geometrically presented as follows:

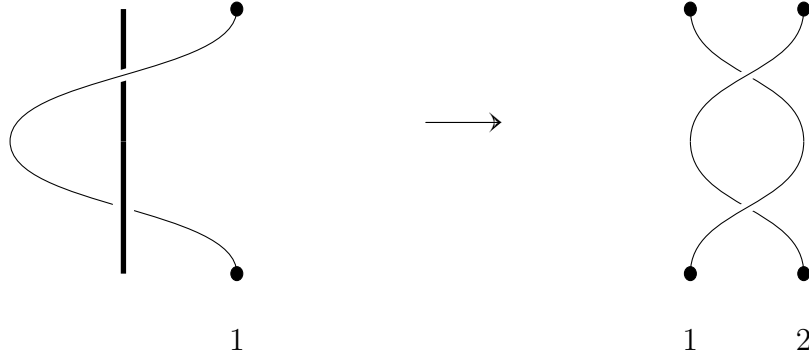


FIGURE 16. $t \mapsto \sigma_1^2$

We see that

$$\begin{aligned} {}_nI(t\sigma_1t\sigma_1) &= \sigma_1^2\sigma_2\sigma_1^2\sigma_2 = \underbrace{\sigma_1\sigma_2\sigma_1}_{\sigma_1}\underbrace{\sigma_2\sigma_1\sigma_2}_{\sigma_2} \\ &= \sigma_2\sigma_1\underbrace{\sigma_2\sigma_1\sigma_2}_{\sigma_2}\sigma_1 = \sigma_2\sigma_1^2\sigma_2\sigma_1^2 = {}_nI(\sigma_1t\sigma_1t). \end{aligned}$$

In other terms ${}_nI$ is a homomorphism. Moreover, it is a monomorphism (see [3]). The following diagram, of injections, is commutative:

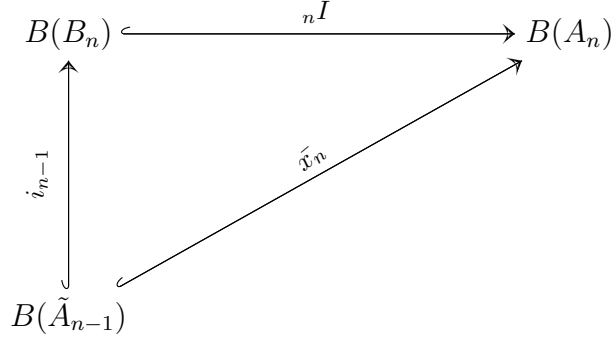


FIGURE 17.

We call \bar{x}_n the composition ${}_nI \circ i_{n-1}$; it is clearly an injection.

We can now subsume our conclusion about the closure of affine braids, which can be viewed as the "affine" version of theorem 5.12.

Corollary 5.13. *Any B-braid, hence any affine one, can be viewed as a braid in some A-type braid group. So, we can define the closure of an affine braid as the closure of its image under \bar{x}_j for some positive integer j . This injection means that any condition forcing any two affine braids to have the same closure is a consequence of the two Markov conditions.*

Proposition 5.14. *Let x be any affine braid in $B(\tilde{A}_{n-1})$ for some $2 \leq n$. Then:*

- 1) *Given y in $B(\tilde{A}_{n-1})$ such that $\phi_n(y) = x$ then $\hat{y} = \hat{x}$ (in other terms $\hat{\cdot}$ is invariant under the action of the Dynkin automorphism).*
- 2) $\widehat{xa_{n+1}} = \hat{x}$.

Proof. Suppose $\phi_n(y) = x$. That is equivalent to saying that $D_n x D_n^{-1} = y$ in $B(\tilde{A}_n)$. But by the first move $D_n x D_n^{-1} \leftrightarrow x D_n^{-1} D_n = x$. Thus $x \leftrightarrow y$.

On the other hand $xa_{n+1} = x\sigma_n^{-1}a_n\sigma_n = xa_n\sigma_n a_n^{-1} \leftrightarrow a_n^{-1}xa_n\sigma_n$, by the first move. But $a_n^{-1}xa_n\sigma_n \leftrightarrow a_n^{-1}xa_n$, by the second move. Hence we are reduced to $a_n^{-1}xa_n \leftrightarrow xa_n a_n^{-1} = x$, by the first move, which means $xa_{n+1} \leftrightarrow x$.

□

Set $\widehat{B(A_n)}$ to be $\{\hat{b}; b \in B(A_n)\}$.

Now we reformulate our description of what we called "B-links", defined as the closures of B-braids. In \mathbb{R}^3 the B-links are those links in which there is an oriented unknotted fixed circle. Now we can talk about $\widehat{B(A_n)}$ defined above without ambiguity (so as for $B(\tilde{A}_n)$).

It is clear geometrically, that considering the fixed circle as a circle among the others gives the way in which $\widehat{B(B_n)}$ is contained in $\widehat{B(A_n)}$, while links in which there is no strings around the fixed circle, gives the inclusion of $\widehat{B(A_{n-1})}$ in $\widehat{B(A_n)}$. It is well known that B -links represent the links in a solid torus, the string which makes the round around the hole of the torus represents \hat{t} or \hat{t}^{-1} , which depends of course on the orientation of the string.

In the same spirit we see that affine links are B -links in which the number of positive rounds equals the number of negative rounds, of course around the fixed circle. Links which do not make rounds are counted here, actually they describe $\widehat{B(\tilde{A}_n)}$ containing $\widehat{B(A_n)}$.

Corollary 5.15. *Suppose that L_1 and L_2 are affine (oriented) links. Suppose that l_1 and l_2 are two affine braids such that $\hat{l}_1 = L_1$ and $\hat{l}_2 = L_2$. Then L_1 and L_2 are isotopic if and only if l_1 and l_2 are equivalent in the sense of Markov, when being viewed in $\cup_{1 \leq i} B(A_i)$, that is: if and only if $\bar{x}_j(l_1)$ and $\bar{x}_j(l_2)$ are equivalent for some j .*

Proof. See theorem 5.2 in [5] with proposition 5.14 .

□

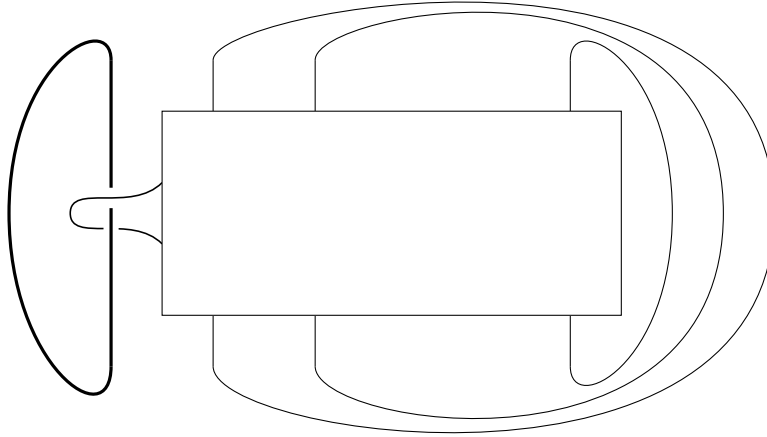


FIGURE 18. Affine closure

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